The Semple tower: a differential approach of curve singularities

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Let Z be a nonsingular algebraic variety of dimension d over an algebraically closed field k of characteristic 0.
In "Some investigations in the geometry of curve and surface elements" (1954), Semple laid the foundations of a tower of projective $\mathbb{P}^{d-1}$- bundles over Z

$$\dots Z(n) \longrightarrow Z(n-1) \dots \qquad \longrightarrow Z(1) \longrightarrow Z$$

His aim was to parametrize the "higher order curvilinear data of Z". In his terminology, this means the chains of infinitely near points of points of Z. In fact, this was not to this question that he gave an answer in this paper.
I heard about the Semple tower for the first time in a series of lectures by J.P. Demailly at the Institute Fourier, Grenoble, in 1996.

§ 1 The inductive step

Let Z be as above and let F be a subbundle of its tangent bundle T_Z; let C be an irreducible and reduced curve in Z.
Definition: C is said to be an integral curve of F if it has the following property: let $\overline{C} \to C$ be the normalization of C and let $\eta : \overline{C} \to C \hookrightarrow Z$; $d\eta : T_{\overline{C}} \to \eta^* T_Z$ factors through $\eta^* F$, ie there exists a commutative diagram

$$\begin{tikzcd} T_{\overline{C}} \arrow[rr] \arrow[dr] & & \eta^* T_Z \\ & \eta^* F \arrow[ur] & \end{tikzcd}$$

We shall construct a nonsingular algebraic variety $\widetilde{Z}$ over Z, a subbundle $\widetilde{F}$ of its tangent bundle $T_{\widetilde{Z}}$ with the same rank as F, a lifting $\widetilde{C}$ of C on $\widetilde{Z}$ which will be an integral curve of $\widetilde{F}$, if C is assumed to be an integral curve of F.
Let $\widetilde{Z} := \mathbb{P}F$; let $\pi : \widetilde{Z} \to Z$; if rank $F = p$, the fibers of π are the projective spaces $\mathbb{P}^{p-1}_k$.
Let x be a nonsingular point of C. By definition, the tangent $T_{C,x}$ of C at x is a line in the fiber F_x of F over x.
Let $\widetilde{x} := (x, [T_{C,x}]) \in \widetilde{Z} = \mathbb{P}F$, we have $\pi(\widetilde{x}) = x$ and a

map $C_{reg} \longrightarrow \tilde{\mathcal{Z}}$, $x \longmapsto \tilde{x}$. The Zariski closure of its image in $\tilde{\mathcal{Z}}$ is a curve $\tilde{C}$ such that $\pi_{|\tilde{C}} : \tilde{C} \to C$.
The points $\tilde{x}$ of $\tilde{C}$ such that $x := \pi(\tilde{x})$ is a singular point of C are the directions of the limits of the tangents to nonsingular points of C whose limit is x; hence they are the directions of the tangents to the branches of C at x.
This follows immediately from L'Hospital's rule:

$$\lim_{t \to 0} \frac{f(t) - f(0)}{g(t) - g(0)} = \lim_{t \to 0} \frac{f'(t)}{g'(t)}$$

$\pi_{|\tilde{C}} : \tilde{C} \to C$ is the <u>Nash blowing-up</u> of C.
On $\tilde{\mathcal{Z}}$, the line bundle $\mathcal{O}_F(-1)$ is a subbundle of $\pi^* F$. Indeed, the fiber of $\mathcal{O}_F(-1)$ at $\tilde{x} = (x, [\ell])$ where ℓ is a line in the fiber F_x of F at $\pi(\tilde{x}) = x$ is the line ℓ.
$\tilde{F}$ is defined to be the subbundle of $T_{\tilde{\mathcal{Z}}}$ such that the diagram

$$\begin{tikzcd}
T_{\tilde{\mathcal{Z}}} \arrow[r] & \pi^* T_{\mathcal{Z}} \\
 & \pi^* F \arrow[u] \\
\tilde{F} \arrow[uu] \arrow[r] & \mathcal{O}_F(-1) \arrow[u]
\end{tikzcd}$$

is cartesian.
We have the commutative diagram of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & T_{\tilde{\mathcal{Z}}/\mathcal{Z}} \arrow[r] & T_{\tilde{\mathcal{Z}}} \arrow[r] & \pi^* T_{\mathcal{Z}} \arrow[r] & 0 \\
0 \arrow[r] & T_{\tilde{\mathcal{Z}}/\mathcal{Z}} \arrow[r] \arrow[u, equal] & \tilde{F} \arrow[r] \arrow[u] & \mathcal{O}_F(-1) \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

Since $\operatorname{rank} T_{\tilde{\mathcal{Z}}/\mathcal{Z}} = p - 1$, we have $\operatorname{rank} \tilde{F} = \operatorname{rank} F - 1 + 1 = \operatorname{rank} F$.
For $x \in C_{reg}$, $\tilde{x} = (x, [T_{C,x}]) \in \tilde{C}_{reg}$, we have

$$d\pi\,(T_{\tilde{C},\tilde{x}}) = T_{C,x} = \mathcal{O}_{F(-1),\tilde{x}}$$

So $T_{\tilde{C},\tilde{x}} \in \tilde{F}_{\tilde{x}}$.
By continuity, we get that $\tilde{C}$ is an integral curve of $\tilde{F}$. We also note that its is not an integral curve of $T_{\tilde{\mathcal{Z}}/\mathcal{Z}}$.

In what follows, our proofs will be based on the choice of "local coordinates" on Z in a neighborhood of a point z and on the charts covering $\widetilde{Z}$.
We choose a regular system of parameters of $\mathcal{O}_{Z,z}$ $(x_1,\dots,x_p;y_1,\dots,y_q)$ such that (F,z) is defined in $T_{Z,z}$ by the equations

$$dy_j = \sum_{1\le i\le p} a_{ji}\, dx_i \qquad 1\le j\le q$$

$(x_1,\dots,x_p)$ will be called "<u>active</u>" coordinates.
Above (Z,z), $\widetilde{Z}$ is covered by charts Z^i, $1\le i\le p$ with coordinates $x_1,\dots,x_p;y_1,\dots,y_q;\ x^i_j = \dfrac{dx_j}{dx_i}$, $1\le j\le p$, $j\ne i$.

$F_i := \widetilde{F}_{|Z^i}$ is defined in T_{Z^i} by the equations $dx_j = x^i_j\, dx_i$, $1\le j\le p$, $j\ne i$ and $dy_j = \sum_\ell a_{j\ell}\, dx_\ell = a_{ji}\, dx_i + \sum_{\ell\ne i} a_{j\ell} x^i_\ell dx_i =$
$(a_{ji} + \sum_{\ell\ne i} a_{j\ell} x^i_\ell)\, dx_i$ for $1\le j\le q$.
On Z^i, the "active coordinates" will be $(x_i, x^i_j\ 1\le j\le p, j\ne i)$, because of the above equations defining F_i in T_{Z^i}.
The terminology was introduced by S. J Colley and G. Kennedy in [C.K] A higher-order contact formula for plane curves Comm. Algebra 19 (2) 1991

§ 2 <u>The Semple tower</u>

The Semple tower

$$\cdots \longrightarrow Z(n) \longrightarrow Z(n-1) \longrightarrow \cdots \longrightarrow Z(1) \longrightarrow Z$$

over Z is defined inductively as follows:
Let $F := T_Z$, $Z(1) := \widetilde{Z}$ defined as in § 1 from Z and F; on $\widetilde{Z}$, we have constructed a subbundle $\widetilde{F}$ of $T_{\widetilde{Z}}$ of rank $d = \operatorname{rank} T_Z = \dim Z$. We set $F(1) := \widetilde{F}$.
Now, from $Z(1)$ and $F(1)$, we construct $\pi(2) : Z(2) := \widetilde{Z(1)} \longrightarrow Z(1)$ and a subbundle $F(2) := \widetilde{F(1)}$ of rank $d = \operatorname{rank} F(1)$.
By iterating this construction, we get the tower

$$\cdots \longrightarrow Z(n) \longrightarrow Z(n-1) \longrightarrow \cdots \longrightarrow Z(1) \longrightarrow Z$$

Each $Z(n)$ is a $\mathbb{P}^{d-1}$-bundle over $Z(n-1)$ and on $Z(n)$, we have a

subbundle $F(n)$ of $T_{\mathcal{Z}(n)}$ of rank d.
From the commutative diagram

$$\begin{array}{ccc} \tilde{\mathcal{Z}} & \xrightarrow{\pi} & \mathcal{Z} \\ \uparrow & & \uparrow \\ \tilde{C} & \xrightarrow{\nu} & C \end{array}$$

where $\nu : \tilde{C} \longrightarrow C$ is the Nash blow-up of the curve C in $\mathcal{Z}$, we get a commutative diagram

$$\begin{array}{ccccccccc} \longrightarrow \mathcal{Z}(n+1) & \xrightarrow{\pi(n+1)} & \mathcal{Z}(n) & \longrightarrow & \cdots & \longrightarrow & \mathcal{Z}(1) & \xrightarrow{\pi(1)} & \mathcal{Z} \\ \uparrow & & \uparrow & & & & \uparrow & & \uparrow \\ \longrightarrow C(n+1) & \xrightarrow{\nu(n+1)} & C(n) & \longrightarrow & \cdots & \longrightarrow & C(1) & \xrightarrow{\nu(1)} & C \end{array}$$

where $\nu(n+1) : C(n+1) \longrightarrow C(n)$ is the Nash blowing-up of $C(n)$ for every $n \geq 0$.
For $n \geqslant 1$, $C(n)$ is an integral curve of $F(n)$, but it is not an integral curve of $T_{\mathcal{Z}(n)/\mathcal{Z}(n-1)}$.

The immersion $T_{\mathcal{Z}(n)/\mathcal{Z}(n-1)} \hookrightarrow F(n)$ gives rise to an immersion

$\mathbb{P}T_{\mathcal{Z}(n)/\mathcal{Z}(n-1)} \hookrightarrow \mathbb{P}F(n) = \mathcal{Z}(n+1)$; we set

$$I_{n+1} := \mathbb{P}T_{\mathcal{Z}(n)/\mathcal{Z}(n-1)}$$

Since $\operatorname{rank} T_{\mathcal{Z}(n)/\mathcal{Z}(n-1)} = d-1 = \operatorname{rank} F_n - 1$, I_{n+1} is a divisor in $\mathcal{Z}(n+1)$, $n \geqslant 1$. It is called the <u>divisor at infinity</u> in $\mathcal{Z}(n+1)$, $n \geqslant 1$.
We shall now introduce the notion of <u>proximity</u> between points of the Semple tower.
For every $n \geqslant 2$, we consider the Semple tower over I_n; it is canonically embedded in the Semple tower over $\mathcal{Z}(n)$; i.e. for every $m \geqslant n$, we have an immersion $I_n(m-n) \hookrightarrow \mathcal{Z}(n)(m-n)$.
It turns out that $I_n(m-n)$ and $\mathcal{Z}(m)$ are two nonsingular varieties which intersect transversally in $\mathcal{Z}(n)(m-n)$.

We set
$$I_n^{+}(m-n) := I_n(m-n) \cap \mathcal{Z}(m)$$
It has codimension $m-n+1$ in $\mathcal{Z}(m)$.

Hence a new tower and a commutative diagram

$$\begin{tikzcd}
{} \arrow[r] & \mathcal{Z}(m) \arrow[r] & \cdots & {} \arrow[r] & \mathcal{Z}(n+1) \arrow[r] & \mathcal{Z}(n) \\
{} \arrow[r] & I_n^{+}(m-n) \arrow[r] \arrow[u, hook] & \cdots & {} \arrow[r] & I_n^{+}(1) \arrow[r] \arrow[u, hook] & I_n \arrow[u, hook]
\end{tikzcd} \qquad \mathcal{S}^{+}_{I_n}$$

<u>Definition</u>: For $m \geq n+1$, we say that $z_m \in \mathcal{Z}(m)$ is <u>proximate</u> to its image $z_n \in \mathcal{Z}(n)$, if either $m = n+1$, or $m \geq n+2$ and $z_m \in I_{n+2}^{+}(m-n-2)$

ie we have

$$\begin{tikzcd}
\overset{z_m}{\mathcal{Z}(m)} \arrow[r] \arrow[rrrrr, dotted, bend left] & \cdots & \mathcal{Z}(n+3) \arrow[r] & \mathcal{Z}(n+2) \arrow[r] & \mathcal{Z}(n+1) \arrow[r] & \overset{z_n}{\mathcal{Z}(n)} \\
I_{n+2}^{+}(m-n-2) \arrow[r] \arrow[u, hook] & \cdots & I_{n+2}^{+}(1) \arrow[r] \arrow[u, hook] & I_{n+2} \arrow[u, hook] & &
\end{tikzcd}$$

<u>Theorem</u>: Let C be an irreducible and reduced curve in $\mathcal{Z}$. For every $n \geq 0$ and every closed point z_n on the n^{th} iterated Nash blowing-up of C, we have
$$m_{C(n), z_n} = \sum_{\{z_m \in C(m)\,;\, z_m \rightarrow z_n\}} m_{C(m), z_m}$$
where $\rightarrow$ is the proximity relation in the Semple tower over $\mathcal{Z}$, and $m_{C(\cdot), z_\cdot}$ is the multiplicity of $C(\cdot)$ at $z_\cdot$.

This is an immediate corollary of

<u>Theorem</u>: Let I_n be the divisor at infinity in $\mathcal{Z}(n)$, $n \geq 2$. Let $(I_n \cdot C(n))_{z_n}$ denote the intersection number of I_n and $C(n)$ at z_n.

Then
$$(I_n \cdot C(n))_{z_n} = m_{C(n-2), z_{n-2}} - m_{C(n-1), z_{n-1}} =$$
$$\sum_{m \geq n,\ z_m \rightarrow z_{n-2}} m_{C(m), z_m}$$

Sketch of proof for $n = 2$, C analytically irreducible at z (ie is a branch) and $\mathcal{Z} = \mathbb{A}^d_k$:

We may assume that the point $z = z_0$ on C is the origin $(0, 0, \dots, 0)$. Let $(x_1, \dots, x_d)$ be a coordinate system on $\mathbb{A}^d_k$.

Let $x_1(t), \dots, x_d(t)$ be the parametrization of C provided by $\eta : \bar{C} \to C \hookrightarrow \mathcal{Z}$

We may assume that $m_{C,z} = \mathrm{ord}_t\, x_1(t)$.

$(C(1), z_1)$ lies on the chart $Z^1 = \mathrm{Spec}\, k[x_1, \dots, x_d; x^1_2, \dots, x^1_d]$ with $x^1_j = \dfrac{dx_j}{dx_1}$, $2 \leq j \leq d$.

Recall that $F^1 := F(1)_{|Z^1}$ is given by the equations $dx_j = x^1_j\, dx_1$, $2 \leq j \leq d$ and that the active coordinates on Z^1 are $(x_1, x^1_2, \dots, x^1_d)$.

Let $\pi(2) : \mathcal{Z}(2) \longrightarrow \mathcal{Z}(1)$; $\pi(2)^{-1}(Z^1)$ is covered by the charts

$Z^{11} = \mathrm{Spec}\, k[x_1, \dots, x_d; x^1_2, \dots, x^1_d; x^{11}_2, \dots, x^{11}_d]$ with

$$x^{11}_j = \frac{dx^1_j}{dx_1}, \quad 2 \leq j \leq d,$$

$Z^{1i_2} = \mathrm{Spec}\, k[x_1, \dots, x_d; x^1_2, \dots, x^1_d; x^{1i_2}_1, \dots, x^{1i_2}_j, \dots, x^{1i_2}_d]_{j \neq i_2}$

with

$$x^{1i_2}_1 = \frac{dx_1}{dx^1_{i_2}}, \dots, x^{1i_2}_j = \frac{dx^1_j}{dx^1_{i_2}}, \quad 2 \leq j \leq d,\ j \neq i_2 \text{ and } 2 \leq i_2 \leq d.$$

On Z^{11} the active coordinates are x_1 and x^{11}_j $2 \leq j \leq d$;

on Z^{1i_2} the active coordinates are $x^1_{i_2}$ and $x^{1i_2}_j$, $1 \leq j \leq d$ $j \neq i_2$.

Since $I_2 = \mathbb{P}T_{\mathcal{Z}(1)/\mathcal{Z}}$, and the active coordinate x_1 on Z^1 is not a new one, we have $I_2 \cap Z^{11} = \emptyset$;

for $2 \leq i_2 \leq d$, $I_2 \cap Z^{1i_2} = \mathrm{div}\, x^{1i_2}_1$, because $x^{1i_2}_1 = \dfrac{dx_1}{dx^1_{i_2}}$ and x_1 was not a new coordinate on Z^1.

Now the parametrization of $C(1)$ on Z^1 is $x_1(t), \dots, x_d(t); x^1_2(t), \dots, x^1_d(t)$ with $x^1_j(t) = \dfrac{x'_j(t)}{x'_1(t)}$, $2 \leq j \leq d$ and $z_1 = (0, \dots, 0; x^1_2(0), \dots, x^1_d(0))$

Therefore

$$m_{C(1), z_1} = \min\left(m_{C,z},\ \mathrm{ord}_t(x^1_j(t) - x^1_j(0))_{2 \leq j \leq d}\right)$$

i) If $m_{C(1), z_1} = m_{C,z}$, we have $(C(2), z_2) \subset Z^{11}$, so $I_2 \cap C(2) = \emptyset$ ie

$$(I_2 \,.\, C(2))_{z_2} = 0 = m_{C,z} - m_{C(1), z_1}$$

ii) If $m_{C(1), z_1} < m_{C,z}$, there exists j with $2 \leq j \leq d$, such that

$$\mathrm{ord}_t\,(x^1_j(t)-x^1_j(0)) < \mathrm{ord}_t\, x_1(t) .$$

We may assume that $j=2$. So $(C(2), z_2)$ lies on the chart $Z^{12} =$ Spec $k[x_1,\dots,x_d; x^1_2,\dots,x^1_d; x^{12}_1, x^{12}_3,\dots,x^{12}_d]$

Since $I_2 \cap Z^{12} = \mathrm{div}\, x^{12}_1$, and $x^{12}_1 = \dfrac{dx_1}{dx^1_2}$, we have

$$x^{12}_1(t) = \frac{x'_1(t)}{x'^1_2(t)}$$

So $(I_2 \cdot C(2))|_{z_2} = \mathrm{ord}_t\, x'_1(t) - \mathrm{ord}_t\, x'^1_2(t) =$

$$= \mathrm{ord}_t\, x_1(t) - \mathrm{ord}_t\,(x^1_2(t) - x^1_2(0))$$

because $x_1(0)=0$ and <u>char $k = 0$</u>.

So

$$(I_2 \cdot C(2))|_{z_2} = m_{C,z} - m_{C(1),z_1} > 0$$

and $z_2 \in I_2$, ie z_2 is proximate to z.

Since the active coordinates on Z^{12} are x^1_2 and x^{12}_j $1 \le j \le d$, $j \neq 2$, we have

$$m_{C(2),z_2} = \min\left(\mathrm{ord}_t\,(x^{12}_j(t) - x^{12}_j(0))_{1\le j\le d, j\neq 2},\ \mathrm{ord}_t\,(x^1_2(t) - x^1_2(0))\right)$$

Let $\pi(3): \mathcal{Z}(3) \longrightarrow \mathcal{Z}(2)$; $\pi(3)^{-1}(Z^{12})$ is covered by charts Z^{12i_3}, $1 \le i_3 \le d$, with new coordinates $x^{12i_3}_j$, $1\le j \le d$, $j \neq i_3$.

If $m_{C(2),z_2} = \mathrm{ord}_t\,(x^{12}_1(t) - x^{12}_1(0))$, $(C(3), z_3)$ lies on the chart Z^{121} on which $x^{121}_2 = \dfrac{dx^1_2}{dx^{12}_1}$, $x^{121}_3 = \dfrac{dx^{12}_3}{dx^{12}_1}, \dots, x^{121}_d = \dfrac{dx^{12}_d}{dx^{12}_1}$

are the new coordinates, since $x^{12}_1, x^1_2, x^{12}_3, \dots, x^{12}_d$ are the active coordinates on Z^{12}.

Now $I_2 \cap Z^{12} = \mathrm{div}\, x^{12}_1$, so $(I_2 \cdot C(2))|_{z_2} = m_{C(2),z_2}$; recall that, by definition, $I^+_2(1) = I_2(1) \cap \mathcal{Z}(3)$, therefore

$$I^+_2(1) \cap Z^{121} = \phi$$

As a consequence z_3 is not proximate to z and we are done.

Next we perform similar computations by induction.

We cover $\mathcal{Z}(n)$ by charts $Z^{i_1\dots i_n}$ such that $\pi(n-1)^{-1}(Z^{i_1\dots i_{n-1}}) = \bigcup_{1\le i_n\le d} Z^{i_1\dots i_n}$

More precisely

$$k[Z^{i_1\dots i_n}] = k[Z^{i_1\dots i_{n-1}}][x_j^{i_1\dots i_n}]_{1\le j\le d,\, j\ne i_n}$$

with
$$x_j^{i_1\dots i_n} = \frac{d x_j^{i_1\dots i_{n-1}}}{d x_{i_n}^{i_1\dots i_{n^b}}} \quad \text{for } j \ne i_{n-1}$$
$$= \frac{d x_j^{i_1\dots i_{(n-1)^b}}}{d x_{i_n}^{i_1\dots i_{n-1}}} \quad \text{for } j = i_{n-1}$$

and $n^b = \inf\{m : 1 \le m \le n,\ i_m = i_{m+1} = \dots = i_n\} - 1$

The active coordinates on $Z^{i_1\dots i_n}$ are the $(d-1)$ new ones $x_j^{i_1\dots i_n}$, $j \ne i_n$ and $x_{i_n}^{i_1\dots i_{n^b}}$.

§ 3 The case $\dim Z = 2$

Theorem 1 : Let (C, z) be a plane branch (ie C is analytically irreducible at $z = z_0$).

Let

$$\longrightarrow C(n) \xrightarrow{\nu(n)} C(n-1) \longrightarrow \quad \dots \quad \longrightarrow C(1) \xrightarrow{\nu(1)} C$$

the sequence of Nash blowing-ups. For $n \ge 1$, let $z_n := \nu(n)^{-1}(z_{n-1})$ ($C(n)$ is a branch at z_n)

Let

$$\longrightarrow C^n \xrightarrow{\sigma_n} C^{n-1} \longrightarrow \qquad \longrightarrow C^1 \xrightarrow{\sigma_1} C$$

the sequence of quadratic transformations, ie for $n \ge 1$, σ_n is the blowing up of z^{n-1} and $\sigma_n^{-1}(z^{n-1}) = z^n$, for $n \ge 1$ and $z^0 = z$.
(C^n is a branch at z^n)

let $m(n)$ (resp. m^n) denote the multiplicity of $C(n)$ (resp. C^n) at $z(n)$ (resp. z^n), $n \ge 1$.

Then

$$m(n) = m^n, \quad n \ge 1.$$

For every $m \ge n+1 \ge 1$, z_m is proximate to z_n if and only if z^m is proximate to z^n (i.e z^m lies on the strict transform of the exceptional divisor of the blowing-up of z^n)

Remarks about the proof: Recall that since (C, z) is a branch $(C(n), z(n))$ and (C^n, z^n) are branches.

let (x_1, x_2) be "local coordinates" on Z on a neighborhood of z.
As above, we cover $Z(n)$ by charts $Z^{i_1 \cdots i_n}$ with $1 \leq i_1, \dots, i_n \leq 2$.
On each one of this charts, we have 2 active coordinates, the new one and a previous one. Let $(C_a(n), \zeta_n)$ be the projection of $(C(n), z(n))$ on the plane whose coordinates are the active ones
First, we easily notice that $m_{C(n), z(n)} = m_{C_a(n), \zeta_n}$.
The proof is by induction on n. The main step is the following lemma.
Lemma: Let (Γ, z) be a plane branch and let $(\Gamma(1), z_1)$ (resp. (Γ^1, z^1)) denote its Nash blowing-up (resp. quadratic transform).
Let Z_i, $i = 1,2$, the two charts covering $\mathbb{A}^2_k(1)$ ie

$$Z^1 = \operatorname{Spec} k\left[x_1, x_2, x_2^1 = \frac{dx_2}{dx_1}\right], \quad Z^2 = \operatorname{Spec} k\left[x_1, x_2, x_1^2 = \frac{dx_1}{dx_2}\right]$$

the active coordinates on Z^1 (resp. Z^2) being x_1 and x_2^1 (resp. x_2, x_1^2).
Let $\alpha^i : Z^i \to \mathbb{A}^2_k$, $i = 1,2$ the projection on the plane of the active coordinates. If $z_1 \in Z^i$, the image $(\Gamma_a(1), \zeta_1)$ of $(\Gamma(1), z_1)$ by α^i and the quadratic transform (Γ^1, z^1) of (Γ, z) are equisingular branches.
Recall that two branches are said to be equisingular if they have the same characteristic exponents.
$(\beta_0, \dots, \beta_g)$ are the characteristic exponents of a branch $(C, 0)$ if $\beta_0 = e_0 = m_{(C,0)}$ and the normalization $\overline{C}$ of C is given by a parametrization:

$$x_1(t) = t^{e_0}$$
$$x_2(t) = \sum_{j=0}^{g} \sum_{i=0}^{s_j} c_{ji}\, t^{\beta_j + i e_j}$$

with $e_{i+1} = \gcd(e_i, \beta_{i+1})$ $0 \leq i < g$ and $e_g = 1$,

$$\beta_{j-1} + s_{j-1} e_{j-1} < \beta_j, \quad 1 \leq j \leq g, \quad s_g = \infty$$

and $c_{j0} \neq 0$, $0 \leq j \leq g$.
The equisingularity of two plane branches is also equivalent to the fact that their sequences of multiplicities of the singular points of their quadratic transformations coincide.
In the proof of the equisingularity of $(\Gamma_a(1), \zeta_1)$ and (Γ^1, z^1) (resp. $(C_a(n), \zeta_n)$ and (C^n, z^n)), we use results of Abhyankar in "Inversion and invariance of characteristic pairs" Amer. J. Math. (1967).

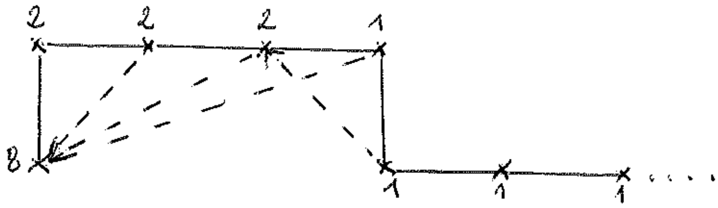
Theorem 2 : Let (C, y) and (D, z) be two plane branches.
For $n \geq 1$, let $(C(n), y_n)$ (resp. $(D(n), z_n)$) the n^{th}-iterated Nash blowing-up of (C,y) (resp. (D,z)); let (C^n, y^n) (resp. (D^n, z^n)) the n^{th} iterated quadratic transform of (C,y) (resp. (D,z)).
We have $y_m = z_m$ for $1 \leq m \leq n$ if and only if $(y^m = z^m)$ for $1 \leq m \leq n$.
As before, the proof uses the projections of $(C(n), y_n)$ and $(D(n), z_n)$ on the plane of their active coordinates. It also uses the "Comparison Lemma" of Abhyankar in "On the semigroup of a meromorphic curve" Proc. Intern. Symp. on Alg. Geometry Kyoto (1978) which computes the intersection multiplicity of two plane branches from their parametrizations.
As a consequence of Th. 1 and Th. 2, we get a 1-1 correspondence between chains of points in the Semple tower over a nonsingular surface S and chains of infinitely near points of S of the same length.

§ 4 An example in dimension three

The following example shows that Theorem 1 and Theorem 2 above do not generalize to branches on nonsingular varieties of dimension 3.
Let C_α in $(\mathbb{C}^3, 0)$ parametrized by

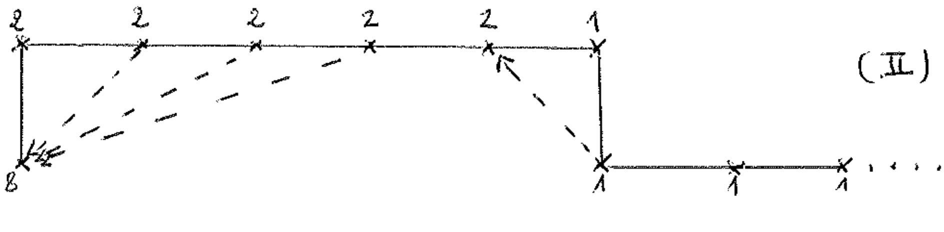
$$x_1 = t^8 \qquad x_2 = t^{10} + t^{13} \qquad x_3 = t^{12} + \alpha t^{15}$$

Recall that the weighted Enriques diagram of a plane branch is a tree with a single branch whose vertices are in 1-1 correspondence with the infinitely near points O_n of O, the edges with pairs $O_{n+1} \to O_n$ and provided with the relation $- - \to$ corresponding to pairs of proximate points; the weight of a vertex is the multiplicity of the corresponding infinitely near point of O.
For $\alpha \neq 2$, this is



(I)

and for $\alpha = 2$,



(II)

The weighted Semple diagram with the proximity relation and the multiplicities in the iterated Nash blowing-ups as in the weighted Enriques diagram is diagram I for $\alpha \neq \frac{52}{25}$ and diagram II for $\alpha = \frac{52}{25}$. So the assertions of Theorem 1 and Theorem 2 do not hold for $\alpha = 2$ and $\alpha = \frac{52}{25}$.

The weighted Semple diagram with the proximity relation and the multiplicities in the iterated Nash blowing-ups as in the weighted Enriques diagram is diagram I for $\alpha \neq \frac{52}{25}$ and diagram II for $\alpha = \frac{52}{25}$.
So the assertions of Theorem 1 and Theorem 2 do not hold for $\alpha = 2$ and $\alpha = \frac{52}{25}$.

§ 4 The Belghitti - Gruson tower

In 1992, H. Belghitti (a doctoral student of L. Gruson) has defined a tower

$$\dots Z^n \longrightarrow Z^{n-1} \dots \longrightarrow Z^1 \longrightarrow Z$$

over a nonsingular algebraic variety Z of dimension d which parametrizes the chains of infinitely near points of points of Z. Here no assumption is required on the characteristic of the base field k.

The construction which will be repeated as an inductive step is the following:

Let $f : X \to S$ be a smooth morphism with X and S nonsingular k-varieties and let $i : S \to X$ be a section of f ie we have $f \circ i = \mathrm{Id}\, S$.

Let $\varphi : \tilde{X} \to X$ be the blowing-up of S in X and let $S_1 := \varphi^{-1}(S)$ be the exceptional divisor of φ in $\tilde{X}$. Let $p_1 : S_1 \to S$ the map induced by φ.

Let $\mathcal{J}$ be the ideal sheaf defining S in X; from the exact sequence

$$0 \longrightarrow \mathcal{J}/\mathcal{J}^2 \longrightarrow \Omega^1_{X/S} \otimes_{\mathcal{O}_X} \mathcal{O}_S \longrightarrow \Omega^1_{S/S} = 0 \longrightarrow 0$$

we get that $S_1 = \mathrm{Proj}\, T_{X/S} \times_X S$; hence $S_1(s) := p_1^{-1}(s) = \mathrm{Proj}\, T_{X(s),s}$ is the set of infinitely near points of s in $X(s) := f^{-1}(s)$.

Now let

$$\begin{tikzcd}
X_1 \arrow[r] \arrow[dd, "f_1"'] & \tilde{X} \arrow[d, "\varphi"] \\
 & X \arrow[d, "f"] \\
S_1 \arrow[r, "p_1"'] & S
\end{tikzcd}$$

be the cartesian diagram.

From the commutative diagrams

$$\begin{tikzcd}
S_1 \arrow[r, hook] \arrow[d, "p_1"'] & \tilde{X} \arrow[d, "\varphi"] \\
S \arrow[r, hook] \arrow[dr, "\mathrm{Id}"'] & X \arrow[d, "f"] \\
 & S
\end{tikzcd}
\qquad
\begin{tikzcd}
S_1 \arrow[r, hook] \arrow[dd, "\mathrm{Id}"'] & \tilde{X} \arrow[d, "\varphi"] \\
 & X \arrow[d, "f"] \\
S_1 \arrow[r, "p_1"'] & S
\end{tikzcd}$$

we get a map $i_1 : S_1 \longrightarrow X_1$ such that $f_1 \circ i_1 = \mathrm{Id}_{S_1}$, ie a section of f_1.

Since $\mathrm{Proj}\, T_{X/S} \to X$ is a smooth map, $p_1 : S_1 = \mathrm{Proj}\, T_{X/S} \times_X S \longrightarrow S$ is also a smooth map, and S being nonsingular, S_1 is also nonsingular. The morphism $f \circ \varphi : \tilde{X} \longrightarrow S$ is smooth, so $f_1 : X_1 \to S_1$ is also smooth, and X_1 is nonsingular.

Hence we have got a smooth morphism $f_1 : X_1 \longrightarrow S_1$ with X_1 and S_1 nonsingular and $i_1 : S_1 \longrightarrow X_1$ a section of f_1.

So we can repeat the process.

We start the building of the tower

$$\dots \mathcal{Z}^n \longrightarrow \mathcal{Z}^{n-1} \dots \longrightarrow \mathcal{Z}^1 \longrightarrow \mathcal{Z}$$

with the smooth morphism $pr_1 : \mathcal{Z} \times \mathcal{Z} \longrightarrow \mathcal{Z}$ and its section the diagonal immersion $i : \mathcal{Z} \longrightarrow \mathcal{Z} \times \mathcal{Z}$

For $z \in \mathcal{Z}$, we have $\mathcal{Z}\times\mathcal{Z}(z) = \mathcal{Z}$, $T_{\mathcal{Z}\times\mathcal{Z}(z),z} = T_{\mathcal{Z},z}$; so for $z \in \mathcal{Z}$, $\mathcal{Z}^1(z) = \mathrm{Proj}\, T_{\mathcal{Z},z}$ is the set of infinitely near points of z in $\mathcal{Z}$ and the tower parametrizes the chains of infinitely near points of points of $\mathcal{Z}$.

Note that $\mathcal{Z}^1 = \mathrm{Proj}\, T_{\mathcal{Z}\times\mathcal{Z}, \mathcal{Z}} = \mathrm{Proj}\, T_{\mathcal{Z}} = \mathcal{Z}(1)$.

Problem: Are $\mathcal{Z}^n$ and $\mathcal{Z}(n)$ isomorphic for $n \geq 2$?