

Aspects of the Monster Tower Construction: Geometric, Combinatorial, Mechanical, Enumerative

Lecture 4: Enumerative Aspects

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- Thanks to Ferran Dachs Cadefau for his preliminary talk.

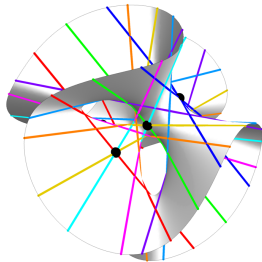
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Enumerative geometry

- Enumerative geometry is a branch of algebraic geometry. It is concerned with problems of counting objects of a specified type, in situations where one expects the answer to be finite.
- We'll give the flavor of the subject via two examples.

- Given a nonsingular cubic surface in \mathbb{P}^3 (the locus of solutions to a degree 3 homogeneous equation in w, x, y, z), how many lines does it contain?
- Answer: 27
- Since this is algebraic geometry, one ought to be asking “Over what field is the surface defined?” The default answer in enumerative geometry is \mathbb{C} , the field of complex numbers.
- But in this problem, the answer is correct even for a real cubic surface: it contains 27 real lines.



(figure by Greg Egan)

- Let C be an algebraic curve of degree d in the complex projective plane. Its *class* is

$$d^\vee := \text{the number of tangents to } C \text{ through a general point of } \mathbb{P}^2.$$

- For a nonsingular curve $d^\vee = d(d - 1)$, but for a singular curve we need correction terms.

- Suppose that C has (ordinary) nodes and cusps, but no more complicated singularities. Then

$$d^\vee = d(d - 1) - 2\delta - 3\kappa,$$

where δ is the number of nodes and κ is the number of cusps.

- If we define two additional quantities

$\beta :=$ the number of bitangent lines to C ,

$\varphi :=$ the number of inflection points (flexes) on C ,

then there are three additional equations:

$$\varphi = 3d(d - 2) - 6\delta - 8\kappa;$$

$$d = d^\vee(d^\vee - 1) - 2\beta - 3\varphi;$$

$$\kappa = 3d^\vee(d^\vee - 2) - 6\beta - 8\varphi.$$

- Collectively, the four equations are known as the *Plücker formulas*.
- According to Kleiman (“The enumerative theory of singularities”), the nonsingular case dates back to a 1756 anonymous work on plane curves, and the idea of having “correction terms” for singularities was apparently due to Poncelet. The formulas above — and the theory of projective duality that facilitates obtaining them — are due to Plücker.

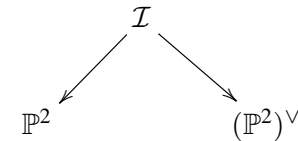
- To discover and prove such formulas, one works with an appropriate *parameter space*.
 - Its points represent objects of a certain type.
 - Moving continuously through the space has the effect of continuously varying the object.
- If you're wondering about lines on a cubic surface, then it's natural to work with the *Grassmannian of lines in \mathbb{P}^3* .
- To understand the Plücker formulas, you want to work with the first monster space over the projective plane, which is also the total space of the projectivized tangent bundle:

$$\mathbb{P}^2(1) = \mathbb{P}T\mathbb{P}^2.$$

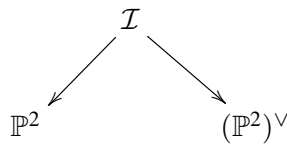
- Classically this space was understood as the *incidence correspondence of points and lines*:

$$\mathcal{I} = \{(p, \ell) : p \in \ell\}.$$

- This point of view reflects an understanding of *projective duality*: the lines in \mathbb{P}^2 are naturally parametrized by another projective plane $(\mathbb{P}^2)^\vee$, and the incidence correspondence naturally maps to both planes.



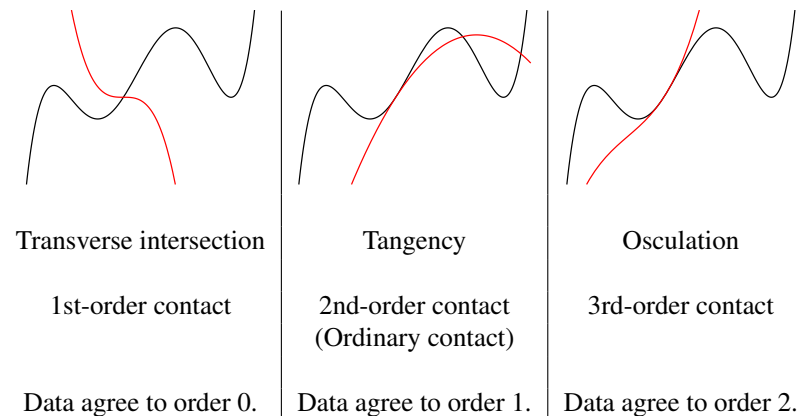
- Given our algebraic curve C in \mathbb{P}^2 , we can lift it up to \mathcal{I} and then project it down to $(\mathbb{P}^2)^\vee$, obtaining its *dual curve C^\vee* .



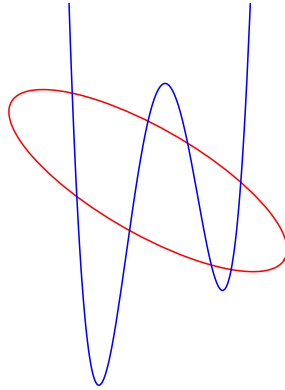
- Under this beautiful duality, the nodes on C correspond to bitangent lines on C^\vee , and vice versa. The cusps on C correspond to flexes on C^\vee , and vice versa. Furthermore, the class of C is the degree of C^\vee , and vice versa.
- Thus the Plücker formulas record an elaborate interplay between the geometry of the two curves, via the intermediary \mathcal{I} .

Contact formulas

- To establish our conventions:



- Bézout's Theorem says that two plane curves of degrees d and e meet in de points.



- To have a completely correct statement, we need to invoke several provisos:
 - We're working over \mathbb{C} .
 - To capture possible "intersections at infinity," we're working in the projective plane.
 - We must assume that the two curves don't have any components in common, e.g., that they don't both contain the same line.
 - If necessary, we count intersections with multiplicity, e.g., a point where the curves are tangent should be counted twice.
- There is a natural generalization to hypersurfaces in projective space \mathbb{P}^n : the intersection of hypersurfaces of degree d_1, d_2, \dots, d_n consists of $d_1 d_2 \cdots d_n$ points; there are similar (somewhat more elaborate) provisos. This is also called Bézout's Theorem.

- Given five general conics in the plane, how many conics are tangent to each of them?
- The first answer to this was given in 1848 by J. Steiner, as follows.
- The equation of a conic

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

uses six coefficients. Thus the parameter space for conics is \mathbb{P}^5 .

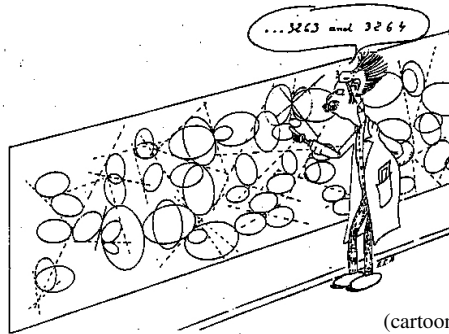
- The condition of being tangent to some specified conic is an equation of degree 6 in A, B, C, D, E, F , i.e., it defines a hypersurface of this degree in \mathbb{P}^5 .
- Since there are five specified conics, this gives us five hypersurfaces.
- According to Bézout's Theorem, they intersect in $6^5 = 7776$ points.

- However, this answer is wrong!
- The parameter space for conics includes all curves of degree 2. Most conics are nonsingular, but some of them are pairs of lines, and, even worse, some of them are just a single line counted twice, e.g.,

$$x^2 + 2xy + y^2 = 0.$$

- If we examine the equation of degree 6 specifying conics tangent to a specified conic, we'll see that all of these double lines satisfy it.
- Thus those five hypersurfaces don't meet in a finite set of points at all! In fact they meet along the 2-dimensional locus of double lines, and at a finite number of additional points.

- This problem was pointed out *circa* 1859 by de Jonquières and in 1864 by Chasles. They also gave the correct count of **3264**.
- We don't want to go into how they did this, except to say — to those who know this notion — that they taught us to use a different parameter space, obtained by blowing up \mathbb{P}^5 along the locus of double lines. This clever construction removes all the intersections except those we really want to count.



(cartoon by Eduardo Casas-Alvero)

- Let Y be a fixed plane curve of degree d , of class d^\vee , and having e flexes.
- Let $\mathcal{X} = \{X_s\}$ denote a 2-parameter family of curves in \mathbb{P}^2 . Define the following *characteristic numbers* for the family \mathcal{X} :

$$M := \# \text{ of } X_s \text{ tangent to a specified line at a specified point on it,}$$

$$K := \# \text{ of } X_s \text{ with a specified general point as cusp,}$$

$$K^\vee := \# \text{ of } X_s \text{ with a specified general line as inflectional tangent.}$$

- In 1880, Schubert gave the following formula for the number of triple contacts between Y and members of \mathcal{X} :

$$dK^\vee + d^\vee K + (3d + e)M$$

- When is this formula valid?
- Hilbert asked the same question, in a much broader way, in the fifteenth of his famous 23 problems of 1900.

The problem consists in this: To establish rigorously and with an exact determination of the limits of their validity those geometrical numbers which Schubert especially has determined on the basis of the so-called principle of special position, or conservation of number, by means of the enumerative calculus developed by him.

For this particular formula, the required hypotheses are quite mild; one should assume:

- Y doesn't contain a line.
- The general member of \mathcal{X} doesn't contain a line.
- \mathcal{X} and Y are in general position with respect to the action of the group $\mathbb{P}GL(3)$ of projective motions of the plane.

A strategy for proving a contact formula

- We're considering a situation involving specified plane curves or families of curves, and asking an enumerative question involving contacts among the curves or members of the family.
- We're looking for a formula whose output is the answer to the question.

- Here's a strategy for finding such a formula:
 - The inputs should be numbers measuring some aspect of the curves or families, called *characteristic numbers*. (In practice, the exact sort of characteristic numbers that are needed, and what they really measure, emerge from the next steps in the strategy.)
 - Construct an appropriate parameter space.
 - Develop its *intersection theory*. (We'll say more about this in a moment.)
 - Apply the intersection theory to the curves or families, obtaining a *proto-contact formula*.
 - Establish the *enumerative significance* of the inputs and outputs. In other words, explain under what conditions they have their intended meanings. Thus the proto-contact becomes an actual contact formula.

- If X is nonsingular, then there is an *intersection ring* A^*X whose product reflects the way in which subvarieties intersect.
- If V and W are nonsingular subvarieties intersecting transversally, i.e., if wherever they meet their tangent spaces span the entire tangent space of X , then

$$[V] \cdot [W] = [V \cap W].$$

Intersection theory

- *Intersection theory* is a type of cohomology theory appropriate to the study of an algebraic variety X .
- The basic objects are certain equivalence classes of subvarieties of X ; linear combinations of these objects are called *algebraic cycle classes*.
- The development of the subject stretches back to the 19th century, and is entwined with the development of homology and cohomology in algebraic topology.

- The intersection ring of n -dimensional projective space is

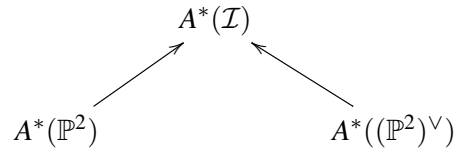
$$A^*(\mathbb{P}^n) = \frac{\mathbb{Z}[h]}{\langle h^{n+1} \rangle}$$

where h is the class of a hyperplane.

- In this ring, the class of a hypersurface of degree d is dh and the class of a point is h^n .
- The ring is graded by codimension, and there is a *degree homomorphism* from the n th graded piece to the integers; it counts, with multiplicity, the number of points, and it's traditionally denoted by \int .
- Here's Bézout's theorem:

$$\int (d_1 h) \cdot (d_2 h) \cdots (d_n h) = d_1 d_2 \cdots d_n.$$

- Recall the incidence correspondence of points and lines in the plane. For the intersection rings, the arrows are reversed: we can “pull back” classes.



- Let $h \in A^*(\mathbb{P}^2)$ be the class of a line. Let $h^\vee \in A^*((\mathbb{P}^2)^\vee)$ be the class of all lines through a specified point of \mathbb{P}^2 (any point). Then

$$A^*(\mathcal{I}) = \frac{\mathbb{Z}[h, h^\vee]}{\langle h^3, (h^\vee)^3, h^2 - hh^\vee + (h^\vee)^2 \rangle}.$$

- For the k th monster over the plane, let i_k denote the cycle class of the divisor at infinity: $i_k = [I_k]$.
- Then

$$A^*(\mathbb{P}^2(k)) = \frac{A^*(\mathbb{P}^2(k-1))[i_k]}{\langle \text{explicit quadratic in } i_k \rangle}.$$

- Suppose we have a variety X carrying a rank two bundle \mathcal{B} . Let $\mathbb{P}\mathcal{B}$ denote the total space of the projectivization of \mathcal{B} . Then

$$A^*[\mathbb{P}\mathcal{B}] \cong \frac{\mathbb{Z}[\varphi]}{\langle \varphi^2 - c_1(\mathcal{B})\varphi + c_2(\mathcal{B}) \rangle}$$

where $c_1(\mathcal{B})$ and $c_2(\mathcal{B})$ are certain cycle classes on X called the *Chern classes* of \mathcal{B} and φ is the *tautological class* $c_1(\mathcal{O}_{\mathbb{P}\mathcal{B}}(1))$.

- This table shows a presentation for $A^*(\mathbb{P}^2(4))$:

\mathbb{P}^2	h	h^3
$\mathbb{P}^2(1)$	h^\vee	$(h^\vee)^2 - hh^\vee + h^2$
$\mathbb{P}^2(2)$	i_2	$i_2(i_2 + 3h^\vee - 3h)$
$\mathbb{P}^2(3)$	i_3	$i_3(i_3 + 3i_2 + 4h^\vee - 5h)$
$\mathbb{P}^2(4)$	i_4	$i_4(i_4 + 3i_3 + 4i_2 + 5h^\vee - 7h)$

A higher-order contact formula

- Here we present one formula of our own, taken from a 1991 paper.
- Suppose
 - Y is a curve in the projective plane.
 - We have a 3-parameter family of plane curves:

$$\mathcal{X} \subset \mathbb{P}^2(3) \times S \rightarrow S$$

- We want to know the number of members of \mathcal{X} having quadruple contact with Y .

- Now where does one look for quadruple contacts? Our answer is: on $\mathbb{P}^2(3)$, the third monster over the projective plane. This is a nonsingular variety of dimension 5.
- The curve Y has a lift $Y(3)$.
- There is also a lift $\mathcal{X}(3) \rightarrow S$; for a general member of the family we just use its lift, but for certain members the lifts may have additional components. We have

$$\mathcal{X}(3) \subset \mathbb{P}^2(3) \times S \xrightarrow{\sigma} \mathbb{P}^2(3).$$

- According to our strategy, we need to work with the intersection ring of $\mathbb{P}^2(3)$. Our aim is to find

$$Q = \int \sigma_*[\mathcal{X}(3)] \cdot [Y(3)],$$

which we feel ought to calculate the number we want.

- In addition to h, h^\vee, i_2 , and i_3 , we work with a class

$$z_3 = [\text{locus of points of } \mathbb{P}^2(3) \text{ representing the data of lines}].$$

- Pairing of $A^1(\mathbb{P}^2(3))$ and $A^4(\mathbb{P}^2(3))$

	$h^2 h^\vee i_2$	$h^2 h^\vee i_3$	$h^2 i_2 i_3$	$(h^\vee)^2 z_3$
h	0	0	0	1
h^\vee	0	0	1	0
i_2	0	1	-3	0
i_3	1	-3	5	0

- Characteristic numbers

	intended meaning
$d := \int h[Y(3)]$	degree of Y
$d^\vee := \int h^\vee[Y(3)]$	class of Y
$k_2 := \int i_2[Y(3)]$	# of cusps on Y
$k_3 := \int i_3[Y(3)]$	# of 3rd-order cusps ($y^2 = x^5$) on Y
	# of members of \mathcal{X} having ...
$\gamma_3 := \int h^2 h^\vee i_2 \cdot \sigma_*[\mathcal{X}(3)]$... cusp at specified point with specified tangent
$\gamma_2 := \int h^2 h^\vee i_3 \cdot \sigma_*[\mathcal{X}(3)]$... 3rd-order cusp at spec.'d point with spec.'d tangent
$\gamma_1 := \int h^2 i_2 i_3 \cdot \sigma_*[\mathcal{X}(3)]$... profound cusp ($y^3 = x^5$) at specified point
$\lambda := \int (h^\vee)^2 z_3 \cdot \sigma_*[\mathcal{X}(3)]$... 3rd-order flex ($y = x^4$) with specified tangent

Orbits and strata

- From these definitions and the inverse transpose matrix, we obtain the proto-contact formula

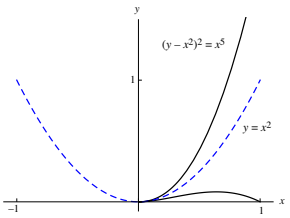
$$Q = d\lambda + d^V\gamma_1 + (3d^V + k_2)\gamma_2 + (4d^V + 3k_2 + k_3)\gamma_3.$$

- When do the quantities appearing in it have their intended meanings? The required assumptions are of these types:
 - Y has no line components, nor does the general member of \mathcal{X} .
 - Restrictions on the types of more complicated singularities, for both Y and the general member of \mathcal{X} .
 - General position with respect to the action of $\mathbb{P}GL(3)$.

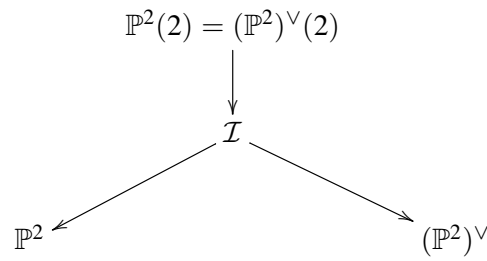
- $\mathbb{P}GL(3)$ acts on $\mathbb{P}^2(3)$, and this action has an open dense orbit.
- In the situation just considered, we say that \mathcal{X} and Y are *suitably transverse* if $\sigma(\mathcal{X}(3))$ and $Y(3)$ meet only in the dense orbit, and if this is a transverse intersection.
- The basic tool for this sort of analysis is *Kleiman transversality theory*. With this theory in hand, we need to simply carry out dimension counts of the intersections of $\sigma(\mathcal{X}(3))$ and $Y(3)$ with the various orbits.

- The eight orbits of $\mathbb{P}^2(3)$ under the action of $\mathbb{P}GL(3)$

Orbit	Stratum	Dimension	Represented by	Parametrization
$\mathcal{O}(0, 0)$	RRR	3	$y = 0$	$x = t, y = 0$
$\mathcal{O}(0, \infty)$	RRV	3	$y^2 = x^5$	$x = t^2, y = t^5$
$\mathcal{O}(0, -)$	RRR	4	$y = x^3$	$x = t, y = t^3$
$\mathcal{O}(\infty, 0)$	RVT	3	$y^3 = x^4$	$x = t^3, y = t^4$
$\mathcal{O}(\infty, \infty)$	RVV	3	$y^3 = x^5$	$x = t^3, y = t^5$
$\mathcal{O}(\infty, -)$	RVR	4	$y^2 = x^3$	$x = t^2, y = t^3$
$\mathcal{O}(-, \infty)$	RRV	4	$(y - x^2)^2 = x^5$	$x = t^2, y = t^4 + t^5$
$\mathcal{O}(-, *)$	RRR	5	$y = x^2$	$x = t, y = t^2$



- The column on the left uses a code we developed in our enumerative work; it describes a finer stratification than the *RVT* stratification that we described in Lecture 2. This stratification takes into account not only the divisors at infinity but also the special locus of data of lines in the plane; since $\mathbb{P}GL(3)$ preserves lines, this locus is likewise preserved. It can also be interpreted as the “dual divisor at infinity.”



Prolongation in families

Level	# of RVT strata	# of $0 - *∞$ strata	# of $\mathbb{P}GL(3)$ orbits
0	1	1	1
1	1	1	1
2	2	3	3
3	5	8	8
4	13	21	21
5	34	55	≥ 56
6	89	144	∞
7	233	377	∞

- Understanding how to lift (prolong) a family of curves is a subtle part of the analysis.
- If you consider a singular curve all by itself, the recipe for lifting says: lift at all nonsingular points, and then take the closure.

- Trying this out on the pair of coordinate axes $xy = 0$, we find that the lift is disconnected: it's the disjoint union of two lines:

$$\begin{cases} y = 0 \\ y' = 0 \end{cases} \text{ (in one chart)} \cup \begin{cases} x = 0 \\ x' = 0 \end{cases} \text{ (in another chart)}$$

- But if this curve is the central member of the family \mathcal{X} consisting of the curves $xy = t$, one should lift it first by lifting all nonsingular members and then taking the closure: this gives the two components we've already seen, together with a third component

$$\begin{cases} x = 0 \\ y = 0 \end{cases}$$

representing the data of all directions over the origin.

- This component appears because, on any nearby curve of the family, we do see all possible directions (except the strictly horizontal and vertical).
- This lift of the central member fits into a nice family $\mathcal{X}(1)$ of lifts, whose other members are just the usual lifts of the other curves in the family.

- For precise work in algebraic geometry, one would like to demand even more: one wants the family to be *flat*. This is a technical condition which gives the best analogue of a continuous family in topology, but it carries more information: each member of the family is a *scheme*, meaning that locally it is cut out by equations.
- The middle component of the lift of $xy = 0$ within the family $xy = t$ is cut out by these equations:

$$\begin{cases} x^2 = 0 \\ xy = 0 \\ y^2 = 0 \\ xy' + y = 0. \end{cases}$$

- For example, suppose one intersects the members of the family with the class h^\vee represented by the set of all lifts of lines through one specified point).
- For a nonsingular member $xy = t \neq 0$ one obtains 2, which we have called the *class* of the curve; it's the number of tangent lines passing through the specified point.
- For the central member $xy = 0$ the intersection occurs on the middle component, and it counts twice.

$$\begin{cases} x^2 = 0 \\ xy = 0 \\ y^2 = 0 \\ xy' + y = 0. \end{cases}$$

- As a point set it's just the line $x = y = 0$.
- But the elements x and y aren't in the ideal we've just described, so as a scheme it's a "thickened line," and in fact one can measure its thickness; it's 2.
- The fact that the family is flat means that when we do intersection calculations with members of the family, the result doesn't suddenly change when $t = 0$.

- Lifting the same family to a higher level, what happens? We analyze this in a 2018 paper. We give an explicit recursive recipe for deriving all the equations of $\mathcal{X}(k)$ in all relevant charts of the monster, and show how they cut out a chain of $2^k + 1$ component curves, each of which is a projective line; we call them *twigs*.
- Here's a picture of $\mathcal{X}(3)$, showing the multiplicities (thicknesses) of each twig. The end twigs are the lifts of the two individual axes; the twig right in the middle is the lift of the component we saw at level 1.

