Aspects of the Monster Tower Construction: Geometric, Combinatorial, Mechanical, Enumerative

Lecture 1: The Monster Tower

Susan Colley Gary Kennedy

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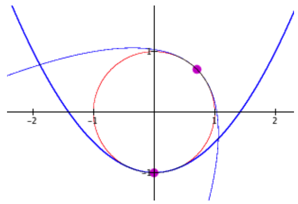
This lecture will explain how three seemingly different situations lead to the same construction:

1. Compactifying curvilinear data (algebraic geometry)
2. Studying Goursat distributions (differential geometry)
3. Analyzing a truck with trailers (dynamics and control theory)

Compactifying curvilinear data

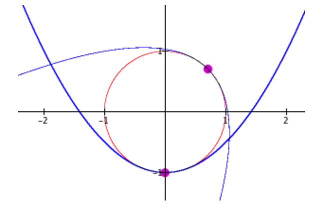
- Start with a smooth manifold or complex manifold or smooth algebraic variety M of dimension $m > 1$.
- Suppose we have two smooth curves C_1 and C_2 passing through a point, and that we have a system of local coordinates $x_1, \ldots, x_m$ based there, so that, for each curve, dx_1 doesn't vanish at the point.
- We say that the curves *have the same curvilinear data* up to order k at the point if the values of all derivatives $d^j x_i/(dx_1)^j$ agree up to order k.

- An example: the parabola $y = \frac{1}{2}x^2 - 1$ and the unit circle



- At the lower point $(0, -1)$:
  - For the parabola
$$\left(y, y', y'', y^{(3)}, y^{(4)}\right) = (-1, 0, 1, 0, 0)\,,$$
  - For the circle
$$\left(y, y', y'', y^{(3)}, y^{(4)}\right) = (-1, 0, 1, 0, 3)\,.$$
- Thus the two curves have the same curvilinear data up to order 3.

- After rotation:



- At the upper point $\left(\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2}\right)$:
  - For the parabola
$$\left(y, y', y'', y^{(3)}, y^{(4)}\right) = \left(\frac{1}{2}\sqrt{2},\ -1, -2\sqrt{2},\ -12,\ -60\sqrt{2}\right)$$
  - For the circle
$$\left(y, y', y'', y^{(3)}, y^{(4)}\right) = \left(\frac{1}{2}\sqrt{2},\ -1,\ -2\sqrt{2},\ -12,\ -72\sqrt{2}\right)$$

- Agreeing in curvilinear data up to order k is $(m-1)k$ conditions, and one can check that they are independent of local coordinates.
- One can see that a *curvilinear datum* is a point in a manifold (or smooth variety) of dimension $m + (m-1)k$. In fact we have a tower of such manifolds, with
  - M as the base,
  - fiber a projective space $\mathbb{P}^{m-1}$ at the first stage,
  - affine space fibers $\mathbb{A}^{m-1}$ thereafter.
- A smooth curve C on M can be *lifted* up through the tower (to whatever order we like). For each point p on C:

$$p \mapsto \text{the point recording its curvilinear data}$$

  The locus of these points is again a smooth curve.

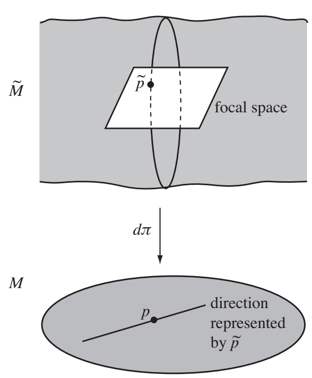
- We have traced this idea back as far as the work of Halphen. In his 1878 doctoral thesis and in a 94-page paper from 1880, he wrote about what we would now call the parameter space for curvilinear data.
- Here are two things we'd like to do:
  1. Give a coordinate-free version of the construction.
  2. Compactify the spaces in the tower, in some natural way.

- The first-order datum is giving us a tangent direction, so at the first stage of the construction we have $M(1) = \mathbb{P}TM$, the total space of the projectivized tangent bundle. The lift of a curve is its *Nash blowup*.
- Naively, one might guess that one just continues in this way: define $M(2) = \mathbb{P}TM(1)$, etc., but this can't be right, because it doesn't give us a manifold of the right dimension. The dimension of $M(2)$ ought to be $3m - 2$, but the dimension of $\mathbb{P}TM(1)$ is $4m - 3$. To say this another way, most points of the proposed space are inaccessible: they can't be reached by lifting (twice) a curve from the base.
- There must be something special about the tangent lines to (once) lifted curves — what is it?
- Semple supplied the answer. To explain what he did, we'll work in a somewhat larger context.

The basic construction

- Suppose that M is a smooth manifold or nonsingular algebraic variety over an algebraically closed field of characteristic 0. Suppose that $\mathcal{B}$ is a rank b subbundle of its tangent bundle TM.
- Let $\widetilde{M} = \mathbb{P}\mathcal{B}$, the total space of the projectivization of the bundle, and let $\pi : \widetilde{M} \to M$ be the projection. A point $\widetilde{p}$ of $\widetilde{M} = \mathbb{P}\mathcal{B}$ over $p \in M$ represents a line inside the fiber of $\mathcal{B}$ at p, and since $\mathcal{B}$ is a subbundle of TM, this is a *tangent direction* to M at p.
- Let
$$d\pi : T\widetilde{M} \to \pi^* TM$$
denote the derivative map of π. A tangent vector to $\widetilde{M}$ at $\widetilde{p}$ is said to be a *focal vector* if it is mapped by $d\pi$ to a tangent vector at p in the direction represented by $\widetilde{p}$; in particular a vector mapping to the zero vector (called a *vertical vector*) is considered to be a focal vector.

- $\widetilde{M} = \mathbb{P}\mathcal{B}$
- $d\pi : T\widetilde{M} \to \pi^* TM$
- A tangent vector to $\widetilde{M}$ at $\widetilde{p}$ is a *focal vector* if it is mapped by $d\pi$ to a tangent vector at p in the direction represented by $\widetilde{p}$.



- The set of all focal vectors forms a subbundle $\widetilde{\mathcal{B}}$ of $T\widetilde{M}$, called the *focal bundle*; its rank is again b.
- Thus we can iterate this construction to obtain a tower of spaces (i.e., smooth manifolds or nonsingular algebraic varieties) together with their associated bundles.

- If we begin the construction by taking $\mathcal{B}$ to be the tangent bundle TM itself, then the resulting tower

$$\cdots \to M(k) \xrightarrow{\pi_k} M(k-1) \xrightarrow{\pi_{k-1}} \cdots \to M(2) \to M(1) \to M(0) = M$$

is called the *Semple tower* over the base M.
- The space $M(k)$ is said to be at *level k*. Observe that it is the total space of a $\mathbb{P}^{m-1}$-bundle over $M(k-1)$.
- $M(1)$ is the total space of the tangent bundle $\mathbb{P}TM$.
- The bundle constructed at step k of the construction is called the *kth focal bundle* and denoted Δ_k — it is a subbundle of the tangent bundle $TM(k)$.

- Gherardelli (1941) — constructed $\mathbb{P}^2(2)$ over the projective plane $\mathbb{P}^2$.
- The full tower was explained by Semple in 1954.
- We learned about the construction from Collino, and used it to study problems of enumerative geometry in three papers in the early 1990s.
- It was treated in greater generality by Lejeune-Jalabert 2006.
- Demailly (1997) used it to study positivity questions for hyperbolic varieties.

Studying Goursat distributions

- Our main source for this exposition is Montgomery–Zhitomirskii, *Geometric approach to Goursat flags* (2001).
- A *distribution* $\mathcal{D}$ on a smooth manifold is a subbundle of its tangent bundle.
- We want to consider *germs* of distributions at a point; thus we may work at the origin in Euclidean space $\mathbb{R}^n$. Two distributions represent the same germ if they agree in some neighborhood of the origin.
- Let d be the rank of $\mathcal{D}$; its *type* is the pair (d, n); the number $n-d$ is called the *corank*.
- There is a natural topology on the set of germs. There is also a natural notion of equivalence of germs: local diffeomorphism taking one distribution to the other.

- How can you tell when two germs are equivalent?
- Here is the well-known theorem of Frobenius:

  A distribution is completely integrable if and only if it is involutive.
- The distribution $\mathcal{D}$ is said to be *completely integrable* if through each point there is a submanifold such that its tangent space there is the fiber of $\mathcal{D}$ at that point.
  - Equivalently, we can find local coordinates $x_1, x_2, \ldots, x_n$ so that at each point $\mathcal{D}$ is spanned by $\partial/\partial x_1, \partial/\partial x_2, \ldots, \partial/\partial x_d$, where d is the rank of $\mathcal{D}$.

- The theorem of Frobenius:

  A distribution is completely integrable if and only if it is involutive.

  To understand the right side, we need the notion of *Lie bracket of vector fields*.
- Suppose that X and Y are vector fields. We can think of them as operators on functions, and thus compose them. We define

$$[X, Y] = XY - YX,$$

  again an operator on functions, and one can prove that it is again a vector field.
- The distribution $\mathcal{D}$ is said to be *involutive* if the Lie bracket of two arbitrary vector fields in $\mathcal{D}$ is again in $\mathcal{D}$.

- Differential geometers and control theorists are interested in noninvolutive distributions.
- It is therefore natural to consider the *Lie square*

$$\mathcal{D}^2 = \mathcal{D} + [\mathcal{D}, \mathcal{D}]$$

  where the second term consists of all possible Lie brackets of vector fields in $\mathcal{D}$. In general this won't be a distribution, since the rank may vary from point to point ...
- ... but let's suppose that it is, and let's iterate the construction to get a sequence of subbundles (as we assume) of the tangent bundle

$$\mathcal{D} = \mathcal{D}_1 \subset \mathcal{D}_2 \subset \mathcal{D}_3 \subset \cdots$$

  where each bundle is the Lie square of the previous bundle. Call this the *Lie squares sequence*.

- We're considering the Lie squares sequence

$$\mathcal{D} = \mathcal{D}_1 \subset \mathcal{D}_2 \subset \mathcal{D}_3 \subset \cdots .$$

- The bundle is said to be *Goursat* if we have this mild growth: the rank increases by one at each step, until we reach the tangent bundle.

- Engel studied Goursat distributions of type $(2, 4)$. Here the sequence of ranks is $2, 3, 4$. He discovered that all such distributions are equivalent. We now call this sole distribution the *Engel distribution*.
- Building on Engel's work, Cartan showed how to *prolong* the Engel distribution to obtain Goursat distributions, and he showed that for a generic prolongation there is a single normal form, i.e., a single equivalence class of Goursat distributions.
  - "Some researchers believe that Cartan missed the singularities [the nongeneric prolongations] in the problem of classifying Goursat distributions. It would be more accurate to say that he was not interested in them." (Montgomery–Zhitomirskii)

- So what is this prolongation procedure?
- In the case $d = 2$, we've already seen it above. If we have a rank 2 Goursat distribution on a manifold M, then we're in the situation of the basic construction, and thus we obtain PM, a $\mathbb{P}^1$ bundle over M, together with a new rank 2 distribution E there.
- The inverse images of the bundles in the Lie squares sequence all increase in rank by one, and E fits in at the beginning:

$$E \subset \pi^* D \subset \pi^* D_2 \subset \cdots .$$

  Thus one observes that again E is Goursat.
- By iterating the construction, one obtains a tower of $\mathbb{P}^1$ bundles over M. Montgomery and Zhitomirskii called it the *monster tower*.

- Now here's what Montgomery and Zhitomirskii did that's really clever: they reversed the construction.
- Suppose D is a germ of a rank 2 Goursat distribution E of corank $s > 2$. They explain how to *deprolong* it to obtain a rank 2 Goursat distribution.
  - The key observation is what they call the Sandwich Lemma.
  - From it, they show how to identify a rank 1 subbundle that ought to be collapsed, so that E collapses down to a rank 2 distribution of corank $s - 1$.
- From this one obtains a beautiful understanding of Goursat distributions of rank 2, as follows: given a germ Goursat distribution of rank 2, deprolong repeatedly until you reach corank 2. You now have the Engel distribution! Thus your given germ can be found somewhere in the repeated prolongation of the Engel distribution.
- In fact one can take this even further, by observing that the Engel distribution can be obtained by twice prolonging the tangent bundle to a surface.

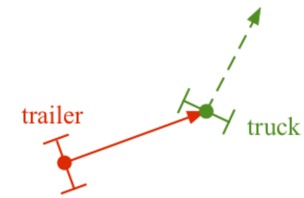
- All of this appears in their 2001 paper and a subsequent monograph.
- Montgomery's student Alex Castro realized that the two tower constructions—Semple and monster—agree. He reported this in his 2010 dissertation at UC Santa Cruz.

- In light of this discovery, the "monster" terminology has now been adopted in greater generality, as a synonym for "Semple." For example, the space $M(k)$ encountered earlier is now called the *kth monster*.
- You might observe that the construction in algebraic geometry is "bottom-up": in trying to analyze curves on a manifold, we are led to construct a tower over it. By contrast, if you are given a Goursat distribution, you are already somewhere up in the tower, and the Montgomery–Zhitomirskii construction shows you how to move downward.

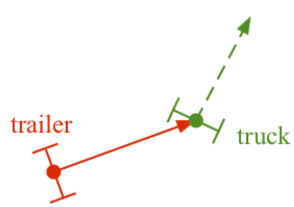
Analyzing a truck with trailers

- Introductory talks about control theory often mention a truck towing a trailer. The truck is free to move in the plane; it can be driven forward or backward, braked, and steered.

- We model the truck as a point with an attached vector, which shows which way it is pointing. The truck can be moved following any differentiable path, subject to the following condition: the velocity vector must point in the direction that the truck is pointing. If the velocity vector is zero, this is interpreted as a vacuous condition (automatically satisfied). It is also legal to drive the truck simply by turning, i.e. by keeping its position fixed while changing which way it's pointing.



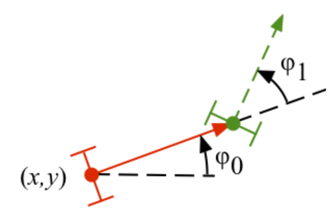
- The trailer is also modeled as a point with attached unit vector, and its motion must satisfy two constraints:
  1. It must be one unit of distance from the truck.
  2. Its velocity must be in the direction of truck. (Again the zero velocity vector always satisfies this condition.)



- In the literature of control theory, the first condition is called a *holonomic constraint*: it's a condition on positions, and it will be incorporated into our choice of configuration space. The second condition is called a *nonholonomic constraint*.

- The *configuration space* consists of quadruples

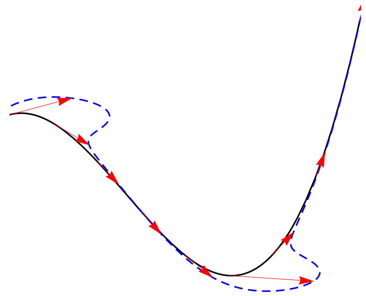
$$(x, y, \varphi_0, \varphi_1).$$



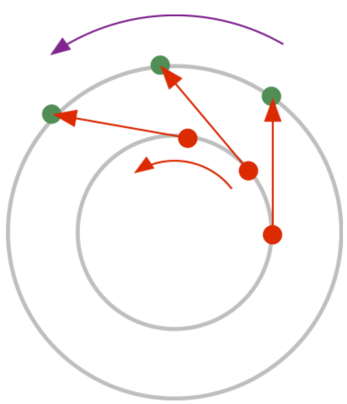
  - (x, y) is the position of the trailer.
  - φ_0 records the direction of the unit vector pointing from the trailer toward the truck (using the positive x-direction as the reference direction).
  - φ_1 records the *bending angle* between the direction of the trailer and the direction of the truck.
- Thus the configuration space is $\mathbb{R}^2 \times S^1 \times S^1$.

- The physical situation suggests, and the mathematical model confirms, that the path of the truck of the truck determines the path of the trailer. In other words, the nonholonomic constraint gives us a differential equation which, given the path of the truck, can be solved to determine the path of the trailer.
- Here are two examples:
    - If the truck and trailer are aligned and we drive the truck forward, the trailer follows in a straight line.
    - We can drive the truck in a unit circle (using the same unit as for the length of the trailer), so that the trailer stays fixed at a point.

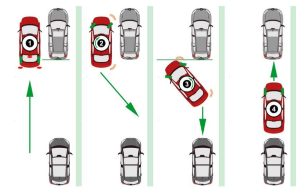
- Control theory is concerned with the opposite direction: if you want the trailer to follow a certain path, how should you drive the truck?
- The answer is a process involving differentiation. There's a nice geometric description: suppose we specify a differentiable path for the trailer. At each point, draw the unit tangent vector, and then mark the head of this vector. The heads of all these vectors trace out the required path of the truck.



- Suppose you want to move the last trailer along a circle of radius r. Then the truck should be moving along the concentric circle of radius $\sqrt{r^2+1}$.



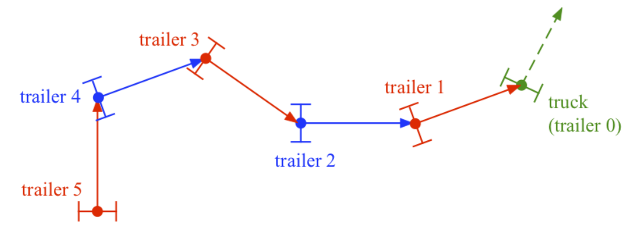
- A related control theory problem: how do you get from one specified configuration to another?
- Consider this problem: you have your automobile lined up in the street parallel to and 3 meters from the sidewalk. next to an empty parking space. The spaces ahead and behind are occupied. You want to move your vehicle to the side, but of course it can't jump to the side. So how do you get it into the space? When we learn to drive a car, the skill of *parallel parking* is one of the bigger challenges.



(from officialdrivingschool.com)

- To reduce the challenge somewhat, let's unhitch the trailer — let's suppose we just have the truck (or that we're riding a unicycle). To maneuver into the space, we should do these four things:
  1. Pull forward.
  2. Turn to face left (assuming the empty space is on the right).
  3. Back up.
  4. Turn rightward (to be facing forward again).
- Mathematically, we've just described a commutator $FTF^{-1}T^{-1}$ of two basic motions. If we have a truck and a trailer, then a similar parsing into basic motions (of the truck) will reveal six individual steps.

- Now let's consider what happens if we add additional trailers: after the first trailer we put a second trailer, following it in the same way that the first trailer follows the truck; and then a third trailer following the second, etc., forming a train.



- In Lecture 3 we'll return to this figure; we'll look at the configuration space and dynamics for trains.

- What we want to observe here is that the position of trailer k is determined from the path of trailer $k+1$ by a process of differentiation: the direction between the trailers is the slope of the tangent to this path.
- This probably sounds familiar! In fact one can show that the configuration space is one of the spaces in the Semple/monster tower over the plane, or nearly so.
- The "nearly so" comes from the fact that the trailers have a direction: they have a front and a back. Thus, to be precise, one needs to redo the basic construction: where previously we said "line" we need to say "ray."
- If one applies this ray-monster construction, then one obtains an unbranched two-sheeted cover $\mathbb{R}^2_{\text{ray}}(1)$ of the usual monster space $\mathbb{R}^2(1)$. If you apply it repeatedly, then you learn that the configuration space for the truck with n trailers is $\mathbb{R}^2_{\text{ray}}(n+1)$, the $(n+1)$st *ray-monster space* over the plane, and that's a 2^{n+1}-sheeted cover of the usual monster $\mathbb{R}^2(n+1)$.